Properties
Equivalence relation
Order relation
N-ary relations

Discrete Mathematics
Relations

(c) Marcin Sydow
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Let $A, B$ be two sets. A **binary relation** between the elements of $A$ and $B$ is any subset of the Cartesian product of $A$ and $B$, i.e. $R \subseteq A \times B$.

We denote relations by capital letters, e.g. $R, S$, etc.

We say that two elements $a \in A$ and $b \in B$ are in relation $R$ iff the pair $(a, b) \in R$ (it can be also denoted as: $aRb$).
Examples

- **empty relation** (no pair belongs to it)
- **diagonal relation** $\Delta = \{(x, x) : x \in X\}$ (it is the “equality” relation)
- **full relation**: any pair belongs to it (i.e. $R = X^2$)
Binary relation can be represented as a predicate with 2 free variables as follows:

Given a predicate $R(x, y)$, for $x \in X$ and $y \in Y$, the relation is the set of all pairs $(x, y) \in X \times Y$ that satisfy the predicate (i.e. make it true)

Each binary relation can be naturally represented as a graph.
$R(x, y)$: “$x$ is less than $y$”

The relation $R$ represented by the above predicate is the set of all pairs $(x, y) \in X \times Y$ so that $R(x, y)$ is true (i.e. $x < y$)
Examples of binary relations

\[ A = B = \mathcal{N} \]

- diagonal relation \( \Delta \ (x = y) \)
- \( x > y \)
- \( x \leq y \)
- \( x \) is a divisor of \( y \)
- \( x \) and \( y \) have common divisor
- \( x^2 + y^2 \geq 10 \)
More examples

Examples of relations on the set $P \times P$, where $P$ is the set of all people.

- $(x, y) \in R \iff x$ is a son of $y$
- $(x, y) \in R \iff x$ is the mother of $y$
- $(x, y) \in R \iff x$ is the father of $y$
- $(x, y) \in R \iff x$ is a grandmother of $y$
More examples

Examples of $R \subseteq P \times C$, where $C$ is the set of all courses in the university for last 5 years.

- $(p, c) \in R \iff p$ passed course $c$
- $(p, c) \in R \iff p$ attended course $c$
- $(p, c) \in R \iff p$ thinks course $c$ is interesting
Domain and co-domain of relation

For binary relation $R \subseteq A \times B$, the set $A$ is called its **domain** and $B$ is called its **co-domain**.

Domain and co-domain can be the same set.
Image and pre-image of relation

**Pre-image** of binary relation \( R \subseteq X \times Y \):
\[
\{x \in X : \exists y \in Y (x, y) \in R\}
\]

**Image** of binary relation \( R \subseteq X \times Y \):
\[
\{y \in Y : \exists x \in X (x, y) \in R\}
\]
Example

\[ A = \{1, 3, 5, 6\}, \ B = \{3, 4, 5, 6, 7\}. \] Relation \( R \subseteq A \times B \) is defined as follows:

\[ xRy \iff x > y \]

\[ R = \{ (5, 3), (5, 4), (6, 3), (6, 4), (6, 5) \} \]

*domain of \( R \): \( A \)
*co-domain of \( R \): \( B \)
*pre-image of \( R \): \( \{5, 6\} \)
*image of \( R \): \( \{3, 4, 5\} \)
Example

\[ A = \{1, 3, 5, 6\}, \quad B = \{3, 4, 5, 6, 7\}. \] Relation \( R \subseteq A \times B \) is defined as follows:

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domain of \( R \)?:
Example

$A = \{1, 3, 5, 6\}$, $B = \{3, 4, 5, 6, 7\}$. Relation $R \subseteq A \times B$ is defined as follows:

$xRy \iff x > y$

$R = \{(5, 3), (5, 4), (6, 3), (6, 4), (6, 5)\}$

domain of $R$?: $A$

coop-domain of $R$?:
Example

\[ A = \{1, 3, 5, 6\}, \quad B = \{3, 4, 5, 6, 7\} \]. Relation \( R \subseteq A \times B \) is defined as follows:

\[ xRy \iff x > y \]

\( R \subseteq \{(5, 3), (5, 4), (6, 3), (6, 4), (6, 5)\} \)

domain of \( R \)?: \( A \)
co-domain of \( R \)?: \( B \)
pre-image of \( R \):
Example

\( A = \{1, 3, 5, 6\}, \ B = \{3, 4, 5, 6, 7\} \). Relation \( R \subseteq A \times B \) is defined as follows:

\[ xRy \iff x > y \]

\( R =? \ \{(5, 3), (5, 4), (6, 3), (6, 4), (6, 5)\}\)

domain of \( R \): \( A \)

codomain of \( R \): \( B \)

pre-image of \( R \): \( \{5, 6\}\)

image of \( R \):
Example

\( A = \{1, 3, 5, 6\}, \ B = \{3, 4, 5, 6, 7\} \). Relation \( R \subseteq A \times B \) is defined as follows:

\( xRy \iff x > y \)

\( R = \{(5, 3), (5, 4), (6, 3), (6, 4), (6, 5)\} \)

domain of \( R \)?: \( A \)

co-domain of \( R \)?: \( B \)

pre-image of \( R \)?: \( \{5, 6\} \)

image of \( R \)?: \( \{3, 4, 5\} \)
Inverse of relation

If $R \subseteq X \times Y$ is a binary relation then its inverse $R^{-1} \subseteq Y \times X$ is defined as $R^{-1} = \{(y, x) : (x, y) \in R\}$

Examples: what is the inverse of:
“$x < y$”? 
Inverse of relation

If \( R \subseteq X \times Y \) is a binary relation then its **inverse**
\( R^{-1} \subseteq Y \times X \) is defined as \( R^{-1} = \{(y, x) : (x, y) \in R\} \)

Examples: what is the inverse of:
“\( x < y \)”?
“\( x \) is a parent of \( y \)”?
Composition of relations

If $S \subseteq A \times B$ and $R \subseteq B \times C$ are two binary relations on sets $A,B$ and $B,C$, respectively, then the **composition** of these relations, denoted as $R \circ S$ is the binary relation defined as follows:

$$R \circ S = \{(a, c) \in A \times C : \exists b \in B[(a, b) \in R \land (b, c) \in S]\}$$

Sometimes it is denoted as $RS$. If $R = S$ then the composition of $R$ with itself: $R \circ R$ can be denoted as $R^2$.

More than 2 relations can be composed. We denote the $n$-th composition of $R$ with itself as $R^n$ (e.g. $R^3 = R \circ R \circ R$, etc.)

Composition is associative, i.e.:

$$(R \circ S) \circ T = R \circ (S \circ T)$$
Example

\[ A = \{0, 1, 2, 3, 4\}, \quad B = \{a, b, c\}, \quad C = \{x, y, z, v\} \]
\[ R = \{(1, a), (2, c), (3, a)\}, \]
\[ S = \{(a, z), (a, v), (b, x), (b, z), (c, y)\} \]
\[ R \circ S = ? \]
Example

\[ A = \{0, 1, 2, 3, 4\}, \quad B = \{a, b, c\}, \quad C = \{x, y, z, v\} \]

\[ R = \{(1, a), (2, c), (3, a)\}, \]
\[ S = \{(a, z), (a, v), (b, x), (b, z), (c, y)\} \]

\[ R \circ S = \?\{(1, z), (1, v), (3, z), (3, v), (2, y)\} \]
Example

\[ A = \{0, 1, 2, 3, 4\}, \quad B = \{a, b, c\}, \quad C = \{x, y, z, v\} \]

\[ R = \{(1, a), (2, c), (3, a)\}, \]

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\[ R \circ S = \{(1, z), (1, v), (3, z), (3, v), (2, y)\} \]

(some join operations in relational databases are based on this operator)
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(some join operations in relational databases are based on this operator)

Is composition commutative?
Example

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\[ S = \{(a, z), (a, v), (b, x), (b, z), (c, y)\} \]

\[ R \circ S = ?\{(1, z), (1, v), (3, z), (3, v), (2, y)\} \]

(some join operations in relational databases are based on this operator)

Is composition commutative? (i.e. is \( R \circ S \) the same as \( S \circ R \) for any binary relations \( R, S \)?)
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A = \{0, 1, 2, 3, 4\}, \ B = \{a, b, c\}, \ C = \{x, y, z, v\}
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R = \{(1, a), (2, c), (3, a)\},
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S = \{(a, z), (a, v), (b, x), (b, z), (c, y)\}
\]

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R \circ S =?\{(1, z), (1, v), (3, z), (3, v), (2, y)\}
\]

(some join operations in relational databases are based on this operator)

Is composition commutative?(i.e. is \(R \circ S\) the same as \(S \circ R\) for any binary relations \(R, S\)?)

For what binary relations their composition is commutative?
The following abstract properties of binary relations are commonly used:

- reflexivity
- symmetry
- counter-symmetry
- anti-symmetry
- transitivity
- connectedness
Reflexivity

Binary relation \( R \subseteq X \times X \) is reflexive iff:
\[
\forall x \in X \ xRx
\]
Examples? (assume \( X \) is the set of all positive naturals)
Binary relation $R \subseteq X \times X$ is reflexive iff:

$$\forall x \in X \ xRx$$

Examples? (assume $X$ is the set of all positive naturals)

“$x$ is a divisor of $y$”? 
Binary relation $R \subseteq X \times X$ is reflexive iff:

$$\forall x \in X \ xRx$$

Examples? (assume $X$ is the set of all positive naturals)
“$x$ is a divisor of $y$”? $x < y$?
Reflexivity

Binary relation $R \subseteq X \times X$ is reflexive iff:

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Examples? (assume $X$ is the set of all positive naturals)
“$x$ is a divisor of $y$”?
$x < y$?
diagonal relation $\Delta$ (i.e. $x \equiv y$)?
Symmetry

Binary relation \( R \subseteq X \times X \) is **symmetric** iff:
\[
\forall x, y \in X \ xRy \implies yRx
\]

Examples? (assume \( X \) is the set of all positive naturals)
Binary relation $R \subseteq X \times X$ is symmetric iff:
\[ \forall x, y \in X \; xRy \Rightarrow yRx \]
Examples? (assume $X$ is the set of all positive naturals)
“$x$ and $y$ have common divisor”?
Symmetry

Binary relation $R \subseteq X \times X$ is **symmetric** iff:

$$\forall x, y \in X \ xRy \implies yRx$$

Examples? (assume $X$ is the set of all positive naturals)

“$x$ and $y$ have common divisor”?

$x \leq y$?
Symmetry

Binary relation $R \subseteq X \times X$ is **symmetric** iff:
\[ \forall x, y \in X \ xRy \implies yRx \]

Examples? (assume $X$ is the set of all positive naturals)
“x and y have common divisor”?
$x \leq y$?
$x == y$?
Counter-symmetry

Binary relation $R \subseteq X \times X$ is **counter-symmetric** iff:

$$\forall x, y \in X \ xRy \Rightarrow \neg(yRx)$$

Examples? (assume $X$ is the set of all positive naturals)
Binary relation $R \subseteq X \times X$ is **counter-symmetric** iff:
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“$x$ and $y$ have common divisor”?
Binary relation $R \subseteq X \times X$ is \textbf{counter-symmetric} iff:

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$x < y$ ?
Counter-symmetry

Binary relation $R \subseteq X \times X$ is counter-symmetric iff:

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Examples? (assume $X$ is the set of all positive naturals)

“$x$ and $y$ have common divisor”?

$x < y$?

$x \equiv y$?
Binary relation \( R \subseteq X \times X \) is anti-symmetric iff:
\[
\forall x, y \in X \ xRy \land yRx \Rightarrow x = y
\]
Examples? (assume \( X \) is the set of all positive naturals)
Anti-Symmetry

Binary relation $R \subseteq X \times X$ is anti-symmetric iff:

$$\forall x, y \in X \ xRy \land yRx \Rightarrow x = y$$

Examples? (assume $X$ is the set of all positive naturals)
“$x$ and $y$ have common divisor”?
Binary relation $R \subseteq X \times X$ is **anti-symmetric** iff:

$$\forall x, y \in X \ xRy \land yRx \Rightarrow x = y$$

Examples? (assume $X$ is the set of all positive naturals)

“$x$ and $y$ have common divisor”?

$x \leq y$?
Binary relation \( R \subseteq X \times X \) is **anti-symmetric** iff:

\[ \forall x, y \in X \ xRy \land yRx \Rightarrow x = y \]

Examples? (assume \( X \) is the set of all positive naturals)

“x and y have common divisor”?

\( x \leq y \) ?

”x is a divisor of y” ?
Transitivity

Binary relation $R \subseteq X \times X$ is transitive iff:

$\forall (x, y, z) \in X, xRy \land yRz \Rightarrow xRz$

Examples? (assume $X$ is the set of all positive naturals)
Transitivity

Binary relation $R \subseteq X \times X$ is transitive iff:

$\forall (x, y, z) \in X, xRy \land yRz \Rightarrow xRz$

Examples? (assume $X$ is the set of all positive naturals)
“$x$ and $y$ have common divisor”? 
Transitivity

Binary relation $R \subseteq X \times X$ is transitive iff:

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Examples? (assume $X$ is the set of all positive naturals)
“$x$ and $y$ have common divisor”?
$x \leq y$?
Transitivity

Binary relation $R \subseteq X \times X$ is **transitive** iff:

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Examples? (assume $X$ is the set of all positive naturals)
“$x$ and $y$ have common divisor”?
$x \leq y$ ?
$x == y$ ?
Closure of a relation

A **closure** of a binary relation $R$ with regard to (wrt) some property $P$ is the binary relation $S$ such that the following conditions hold:

- $S$ has the property $P$
- $R \subseteq S$ (S “extends” $R$)
- $S$ is the smallest (with regard to inclusion) relation satisfying the two above conditions (i.e. for any $T$ such that $R \subseteq T$ it holds that $S \subseteq T$.

The property $P$ can be for example: transitivity, symmetry, reflexivity, etc.

Notice: the closure of relation may not exist (example?:)
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The property $P$ can be for example: transitivity, symmetry, reflexivity, etc.

Notice: the closure of relation may not exist (example?: a counter-symmetric closure of a symmetric relation, etc.)
Examples: How to compute the closure of a relation?

If $R$ is a binary relation, let's consider how to compute its reflexive, symmetric and transitive closure:

- reflexive closure of $R$
Examples: How to compute the closure of a relation?

If $R$ is a binary relation, let's consider how to compute its reflexive, symmetric and transitive closure:

- reflexive closure of $R$? $R \cup \Delta$
- symmetric closure of $R$?
Examples: How to compute the closure of a relation?

If $R$ is a binary relation, let's consider how to compute its reflexive, symmetric and transitive closure:

- reflexive closure of $R$? $R \cup \Delta$
- symmetric closure of $R$? $R \cup R^{-1}$
- transitive closure of $R$?
Examples: How to compute the closure of a relation?

If $R$ is a binary relation, let's consider how to compute its reflexive, symmetric and transitive closure:

- reflexive closure of $R$? $R \cup \Delta$
- symmetric closure of $R$? $R \cup R^{-1}$
- transitive closure of $R$? $R \cup R^2 \cup R^3 \ldots = \bigcup_{i \in \mathbb{N}^+} R^i$
Examples: transitive closure of relation

For a binary relation $R \subseteq X^2$ its **transitive closure** is defined as the smallest relation $T$ so that $T$ is transitive and $R \subseteq T$

Example: transitive closure of:

"x is a son of y"?
"x == y"?
"x \geq y"?
Examples: transitive closure of relation

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Examples: transitive closure of relation

For a binary relation $R \subseteq X^2$ its **transitive closure** is defined as the smallest relation $T$ so that $T$ is transitive and $R \subseteq T$

Example: transitive closure of:
“x is a son of y”?  
“$x == y$”?  
“$x \geq y$”?
A binary relation $R \subseteq X^2$ is equivalence relation iff it is:

- reflexive
- symmetric
- transitive
Examples? (assume $X$ is the set of all positive naturals)
Examples? (assume $X$ is the set of all positive naturals)

$x \equiv y$?
Examples? (assume $X$ is the set of all positive naturals)

$x == y$?

“$x$ and $y$ have common divisor”?
Examples? (assume $X$ is the set of all positive naturals)

$x \equiv y$?

“$x$ and $y$ have common divisor”?

$x \leq y$?
Examples? (assume $X$ is the set of all positive naturals)

$x \equiv y$?
“$x$ and $y$ have common divisor”?
$x \leq y$?
“$x-y$ is even”?
An **equivalence class** of the element $x \in X$ of the equivalence relation $R \subseteq X^2$ is defined as:

$$[x]_R = \{ y \in X : xRy \}$$

(notice that, due to symmetry of equivalence relation, $xRy$ is equivalent to $yRx$)

For $[x]_R$, $x$ is called the **representative** of this equivalence class.

There can be many representatives of the same equivalence class.
Partition of a set

A family $F$ of non-empty subsets of some set $X$ is called a **partition** of $X$ if the following two conditions hold:

- for any two different $A, B \in F$ it holds that $A \cap B = \emptyset$
- $X$ is the union of all sets from $F$ ($X = \bigcup F$)

Each set from $F$ is called a **partition block**.

Examples?
Partition of a set

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- for any two different $A, B \in F$ it holds that $A \cap B = \emptyset$
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Each set from $F$ is called a **partition block**.

Examples?
odd and even numbers form two blocks of partition of integers
Properties of equivalence classes

If \([x]_R\) and \([y]_R\) are two equivalence classes of some equivalence relation \(R\), then either:

- \([x]_R \cap [y]_R = \emptyset\) (do not intersect)
- or:
  - \([x]_R = [y]_R\) (are identical)

Since \(\forall x \in X\ [x]_R \neq \emptyset\) (due to reflexivity of \(R\)), and different equivalence classes are disjoint the following holds:

The equivalence classes **partition** the domain of the equivalence relation.
Example

What are the equivalence classes of the following equivalence relations?

- $x == y$
- “$x$ has the same diploma supervisor as $y$”
Quotient of the set by equivalence relation $R$ (operation of abstraction)

Given an equivalence relation $R \subseteq X^2$ we call the family of all its equivalence classes the **quotient of $X$ by $R$**:

$$X/R = \{[x]_R : x \in X\}$$

(the similarity to division symbol for numbers is not coincidental, since it has some similar properties)

The $X/R$ operation is also called the “abstraction operation”, i.e. we abstract from any properties that are indifferent for the equivalence relation $R$. 
Example

What is $X/R$ if:

- $X$ is the set of natural numbers and $R$ is equality ($x = y$)?
- $P$ is the set of students and $R$ is the set of pairs of students that have the same diploma supervisor?
Example

What is $X/R$ if:

- $X$ is the set of natural numbers and $R$ is equality ($x = y$)?
Example

What is $X/R$ if:

- $X$ is the set of natural numbers and $R$ is equality ($x = y$)?
- $P$ is the set of students and $R$ is the set of pairs of students that have the same diploma supervisor?
Consider a relation \( R \subseteq X^2 \) is called a \textbf{partial order} and four properties:

1. reflexive
2. anti-symmetric
3. transitive
4. \( \forall x, y \in X \ xRy \lor yRx \)

Relation \( R \) is:

- partial order if it satisfies conditions 1-3 above
- quasi order if it satisfies only 1 and 2
- linear order if it satisfies all conditions 1-4 above
Is the following relation a partial order, quasi order, linear order,?

- \(\leq\) (on numbers)?
- \(\Delta\) (on any set)? (“\(x=y\)’’)
- \(<\) (on numbers)
- \(\subseteq\) (on sets)?
An n-ary relation $R$, for $n \in \mathbb{N}$ is defined as $R \subseteq X_1 \times X_2 \ldots X_n$. Binary relation is a special case for $n = 2$. In particular, for:

- $n = 1$, 1-ary relation is the set of some elements of the domain that satisfy some property (e.g. even numbers, etc.)
- $n = 0$, 0-ary relation, that is empty can be theoretically interpreted as a constant in the domain of the relation (e.g. “0” in natural numbers) that has some special properties
Example tasks/questions/problems

For each of the following: precise definition and ability to compute on the given example (if applicable):

- Relation and basic concepts
- Properties of binary relations
- Composition and inverse
- Equivalence relation, equivalence classes
Thank you for your attention.